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ON SUPERSONIC COMPRESSORS AND NOZZLES

(Supplement to AMP Report 82.1R)

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Prepared for the
APPLIED MATHEMATICS PANEL
NATIONAL DEFENSE RESEARCH COMMITTEE
By the
Applied Mathematics Group
New York University

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INTRODUCTION

In response to various inquiries and suggestions concerning the report "THEORETICAL STUDIES ON THE FLOW THROUGH NOZZLES AND RELATED PROBLEMS" (AMP Report 82.1R), asking for supplementary information, the following report was prepared ~~for~~ by Dr. K. O. Friedrichs of the Applied Mathematics Group, New York University, with the assistance of Dr. C. DePrima, Miss M. Johnson, and Mrs. B. Grossman. The present supplementary report describes in detail the construction of a perfect nozzle and contains some additional remarks concerning the accuracy and the theoretical background of the construction described in the previous report.

R. Courant

Technical Representative
Contract OEMsr-945

ON SUPERSONIC COMPRESSORS AND NOZZLES

In Appendix II of the previous report a general method was described to determine contours for a compressor duct (with rotational symmetry) into which a supersonic stream of air with constant velocity may enter and be adiabatically compressed without interference of shocks. In the meantime this method has been carried out in a simple case and the results are now presented in this supplementary report.

In the main part of the previous report a method was described to determine isentropic flows through nozzles with rotational symmetry by employing an expansion whose terms of order zero agree with the results of the hydraulic treatment. Only terms of first and second order were then considered for the refined treatment. To check the accuracy of this procedure, terms of third and fourth order have now been taken into account in a simple case. The results are presented in the second section of this supplementary report.

Furthermore, this supplementary report contains, in its third section, a more detailed derivation of the basic differential equations employed.

1. Construction of a Supersonic Compressor or Duct.

For the construction of a compressor duct it has been assumed that viscosity and heat conduction may be neglected. In the actual design of a compressor duct this assumption can no longer be upheld and for this reason the design presented here can be considered only an approximation.* Nevertheless, this approximation probably gives a very good indication of the proper kind of contour shapes, and in particular of the length of the converging section of the duct for a prescribed compression ratio.

Since viscosity and heat conduction are ignored, the flow through a supersonic compressor duct is the reverse of the flow through a perfect nozzle (a nozzle being called perfect if it produces an exhaust flow with constant velocity). For convenience we shall in the following refer to exhaust flow rather than to compressor flow.

The existence of perfect three-dimensional nozzle flows had been indicated by A. Busemann [15], 1942,** who, however, did not describe a construction. It had escaped the attention

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* Also for another reason the presented design is only an approximation, viz., because shocks must be admitted in the flow since otherwise the flow would be unstable.(See Section 9 in the previous report.)

** Reference numbers refer to the bibliography of the previous report.

of the writer of the previous report that a construction of a perfect nozzle had already been described earlier by F. J. Frankl [21], 1934. A detailed outline of the construction of a "Frankl nozzle" (consisting of 22 steps) is described by M. S. Kisenko, (see bibliography at the end of this report).

Other references concerning nozzle flow which in the meantime came to the attention of the writer are listed in the bibliography.

The design described in the present report employs the steps outlined in Appendix II, Section 13 of the previous report with minor modifications which will be indicated.

We denote the pressure by p, the density by ρ, the flow speed by q, the sound speed by c, the Mach number q/c by M, the cross-section area by A. The subscript c refers to the chamber, * to the throat, e to the mouth.

A pressure ratio $p_c/p_e = 5.66$ was assumed;* it corresponds to the expansion ratio $A_e/A_* = 1.43$ and the exhaust Mach number $M_e = 1.79$. See the first line in Table I.

A regular non-perfect exhaust flow F_o was constructed by the method of Part I of the previous report. The point A_o on the axis was found at which the desired Mach number M_e was reached. A Mach line C_o in backward direction through

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* A value somewhat less than 6 was desired; the value 5.66 was convenient for numerical computation.

A_o was determined. (See Section 7 of the previous report.) The flow F_o was cut off beyond the transition line C_o, and there replaced by a flow G which is turned over into a parallel flow (see Sections 7 and 13). The results are shown in Table II. This figure exhibits firstly the streamlines η = const. The quantity η is the distance of the streamline from the axis at the throat. The quantity $\pi \rho_* q_* \eta^2$ is then the flux of mass per unit time through the tube bounded by the surface η = const.

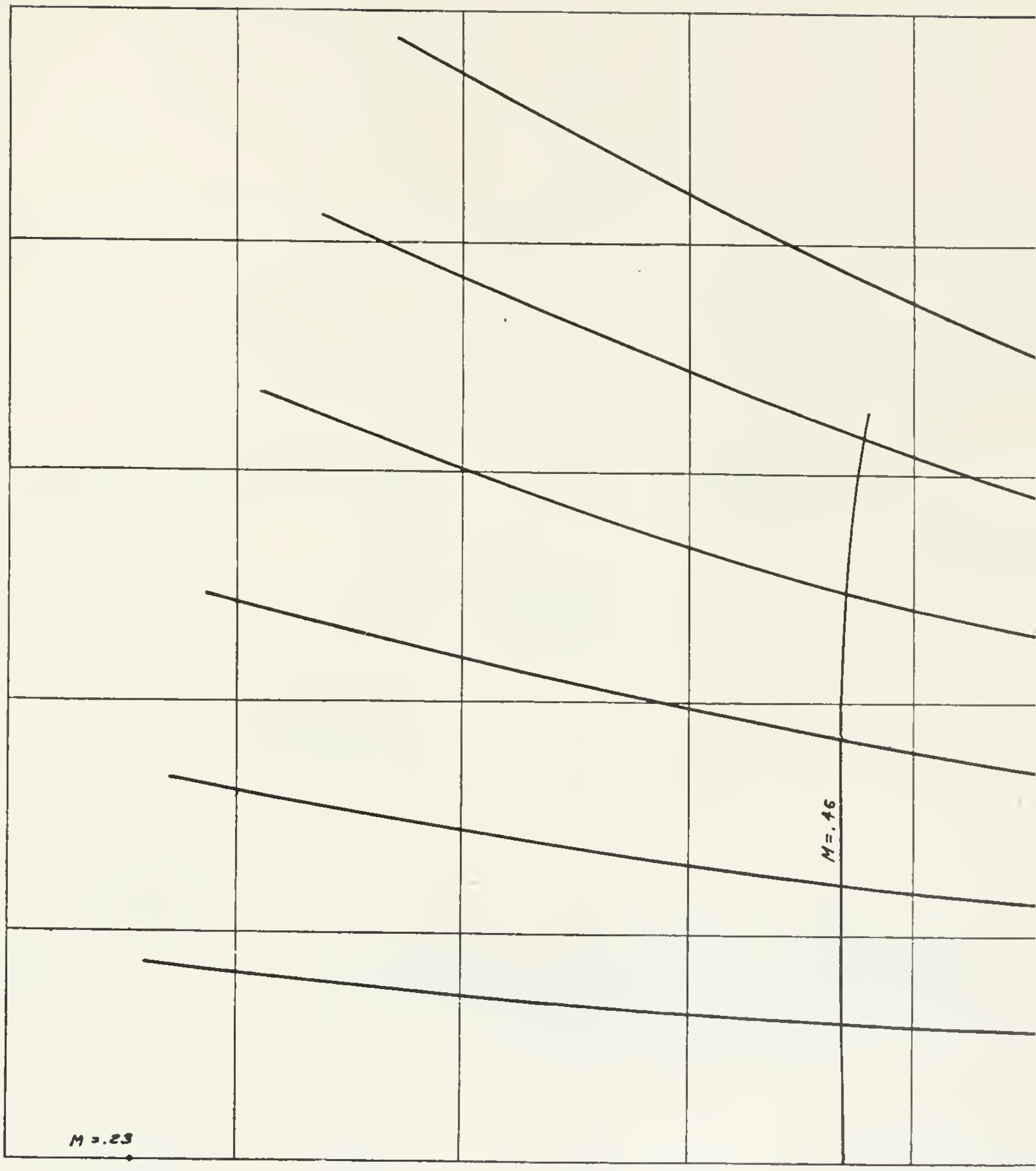
Further, the lines q = const. or what is equivalent, p = const. or M = const. are shown. They are labeled by the values of the Mach number M along these lines. The corresponding values of p_c/p and other relevant quantities constant on these lines are given in Table I. The line M = 1 separates the region of subsonic and supersonic flow. The line M = 1.79 is the straight Mach line D_o starting at the point A_o on the axis beyond which the flow has constant velocity q_e in the axial direction.

Finally, a set of Mach lines is drawn in the supersonic range.

One observes that the backward Mach lines C (and also the lines M = const.) intersect each other near the throat cross-section for about η = .7. Continued, these lines would have a cusp-like envelope, a "limiting line"; and the field of flow would fold over. Clearly, the flow could not be used beyond the streamline η = .6 and even the location of the line

TABLE I

		$\frac{P}{P_*}$	$\frac{q}{q_*}$	$\frac{T}{T_*}$	$\frac{A}{A_*}$	$\frac{P}{P_c}$	$\frac{P_c}{P}$	$\frac{q}{q}$	$\frac{T}{T_c}$
Mouth	1.79	.33	1.53	.73	1.43	.18	5.66	.625	.61
	1.67	.40	1.47	.77	1.314	.21	4.76	.6	.64
	1.56	.46	1.41	.80	1.23	.25	4.08	.575	.67
	1.48	.54	1.35	.84	1.16	.28	3.53	.55	.70
	1.44	.57	1.32	.85	1.14	.30	3.34	.54	.71
	1.35	.61	1.29	.87	1.10	.32	3.09	.525	.72
	1.30	.69	1.23	.90	1.06	.37	2.74	.5	.75
	1.024	.85	1.10	.96	1.02	.45	2.22	.45	.80
Throat	1.00	1.00	1.00	1.00	1.00	.53	1.89	.4082	.83
	.98	1.03	.98	1.01	1.00	.54	1.84	.4	.84
	.70	1.36	.74	1.09	1.09	.72	1.39	.3	.91
	.46	1.65	.49	1.16	1.43	.87	1.15	.2	.97
	.23	1.83	.25	1.19	1.63	.97	1.04	.1	.99
Chamber	0	1.89	0	1.20		1.00	1.000	0	1.00


M = .23
M = .46

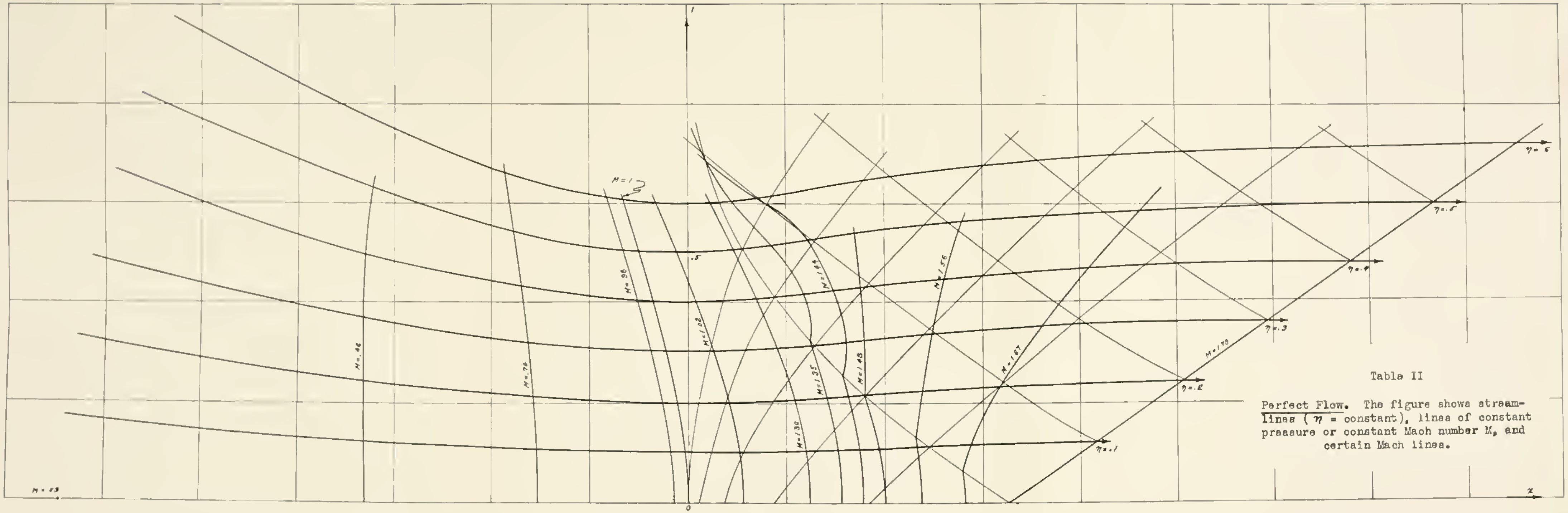


Table II

Perfect Flow. The figure shows streamlines (η = constant), lines of constant pressure or constant Mach number M, and certain Mach lines.

η = .6 is not quite certain.

Each of the streamlines shown in the figure may serve as a nozzle contour. For the purpose of comparison these different streamlines were so magnified that they correspond to the same throat diameter and then superimposed. The result is shown in Table III. The coordinates of points on these contours corresponding to throat, transition, and exit are given in Table IV; R being the radius of curvature at the throat. The contour corresponding to η = .1 is omitted because it would be too long; even the contour η = .2 is rather long. It appears reasonable to employ the line η = .5 for design. (The line η = .6 is not much shorter and its location is uncertain.) The contour η = .5 yields a length of 2.5 throat diameters between throat and exit, a length which appears rather acceptable.

As to the details of the calculation, the following may be mentioned. The basic flow F_o was chosen as one whose streamlines are essentially hyperbolas; more exactly $h_o=\sqrt{1+\xi^2}$ in the terminology of Part I of the previous report. The streamlines were drawn according to formulas (2.11) and (2.12), the latter formula being simplified to $y = h_o\eta$ since the third order term turned out to be very small.

The Mach line C_o was drawn from the differential equation (see 13.09 and 13.10)

$$dx + \tan(\beta + \theta)dy = 0,$$

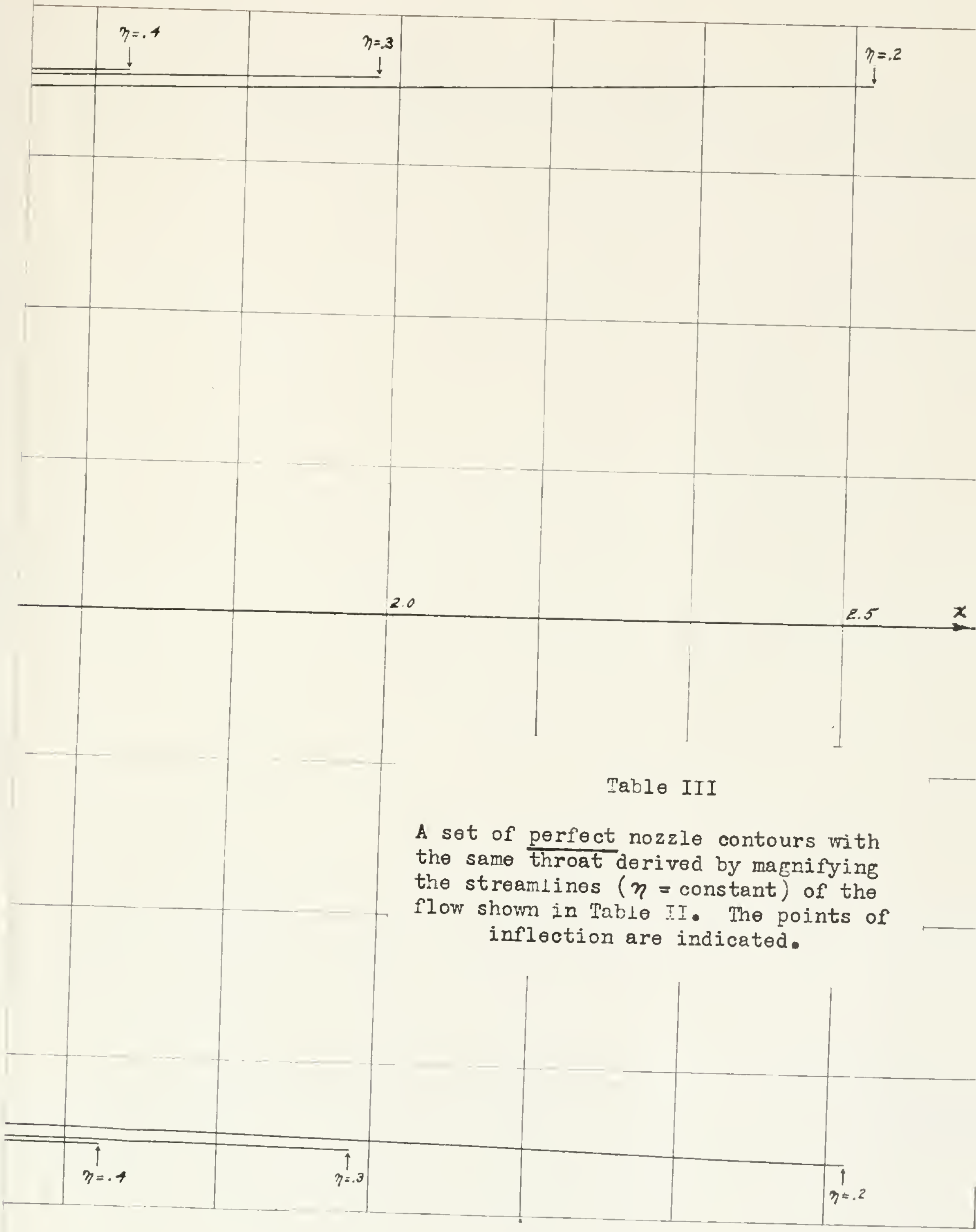


Table III

A set of perfect nozzle contours with the same throat derived by magnifying the streamlines (η = constant) of the flow shown in Table II. The points of inflection are indicated.

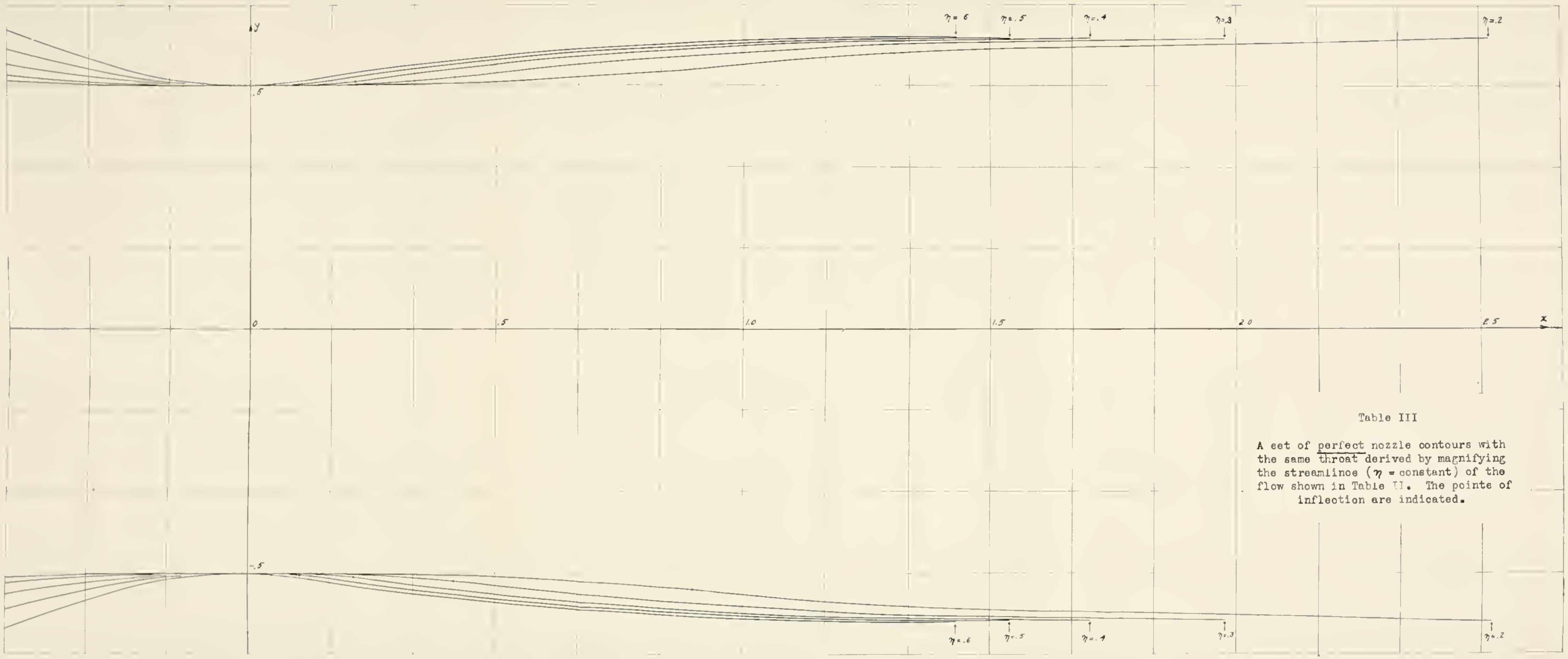


Table III

A set of perfect nozzle contours with the same throat derived by magnifying the streamlines (η = constant) of the flow shown in Table II. The points of inflection are indicated.

TABLE IV

	.1			.2			.3			.4			.5			.6		
	x	y	R	x	y	R	x	y	R	x	y	R	x	y	R	x	y	R
Throat	0	.5	10	0	.5	5	0	.5	3.33	0	.5	2.5	0	.5	2	0	.5	1.67
Transition	2.46	.56	--	.90	.53	-	.42	.52	---	.22	.51	---	.11	.50	-	.05	.50	--
Mouth	4.08	.60	--	2.52	.60	-	2.98	.60	---	1.70	.60	---	1.54	.60	-	1.43	.60	--

Coordinates of rim and point of transition for the contours shown in Table III in terms of the throat diameter as unit.

R is the radius of curvature at the throat.

$\beta = 90^\circ - \alpha$ here being the angle which was called η in Section 13, and α being the Mach angle. Expansions of x, y, θ, β with respect to η were inserted and terms of second and higher order were neglected. The result was the equation

$$d\xi + h_o(\xi)\tan\beta_o(\xi)d\eta = 0.$$

Here $90^\circ - \beta_o$ is the Mach angle corresponding to $q/q_* = q_o(\xi)$; hence $\cos\beta_o = M_o$ can be expressed in terms of $q_o(\xi)$.

The equation leads to the integral

$$\eta = -\int_{\xi_o}^{\xi} \frac{d\xi}{h_o(\xi)\tan\beta_o(\xi)}$$

which could be evaluated. Since, however, just the points with $\eta = .1, .2, ..$ were desired in particular, a finite difference procedure was employed with the step $\Delta\eta = .1$. The Mach line C_o could then be drawn.

The flow denoted by G in Section 7 could then be determined by the method indicated in Section 13. The only modification was that the angle β (there called η) as a function of ξ was not read off from Busemann's diagram but obtained by inversion from a table giving ξ as a function of β easily obtained on the basis of (13.03) and (13.02).

We chose the parameters σ and τ (see Fig. 16 in the previous report) so that they were identical with the quantity η on the Mach lines D_o and C_o respectively; $\Delta\sigma = \Delta\tau = .1$ was chosen as the step. The advantage of doing so was that then the intersections of the streamlines $\eta = .1, .2, ..$ with these Mach lines were just the net points on them.

The procedure indicated in Section 13 then proved to be manageable. Near the axis it might perhaps have been simplified since it turned out that the Mach lines D were nearly straight there. The directions of these lines on the Mach line C_o being given from the main flow F_o, one might have drawn these Mach lines straight and then proceeded in the same manner as for the corresponding two-dimensional construction.

2. The Accuracy of the Determination of Nozzle Flow by the Method of Expansion.

To obtain some information on the accuracy of the method for determining the nozzle flow explained in Part I of the previous report, the expansion has now been carried through up to the fourth order in a simple example and the results have been compared with those obtained if only terms up to the second order are considered.

The example taken is the same as that used for the basic flow F_o in the construction of the preceding section, viz.:

$$h_o = \sqrt{1 + \xi^2}$$

yielding streamlines which are approximately hyperbolas. The results of drawing streamlines from

$$x = \xi + x_2\eta^2 + x_4\eta^4, \; y = yh_o\eta\left[1 + y_2\eta\right]$$

are compared with those resulting from

$$x = \xi + x_2\eta^2 \quad , \quad y = h_o\eta$$

in Table V. It is seen that deviations are noticeable for $\eta = .5$ and $\eta = .6$. Up to $\eta = .4$ the first approximation might be considered sufficient.

The coefficients x_2, y_2 are given in (2.11) and (2.12). The coefficient x_4 is

$$x_4 = -\frac{1}{32} h_o^2 \left\{ (5Q_o + 2)\, h_o' h_o'' + Q_o' h_o h_o'' + Q_o h_o h_o''' \right\} ,$$

with

$$Q_o = M_o^2 - 1 ,$$

$$Q_o' = 4\mu^2 \frac{h_o'}{h_o} \frac{Q_o + 1}{q_o^2 - \mu^2} .$$

For $h_o(\xi) = \sqrt{1 + \xi^2}$, in particular

$$x_4 = -\frac{1}{16} \frac{\xi}{1 + \xi^2} M_o^2 \frac{(q_o/q_*)^2 + 1}{(q_o/q_*)^2 - 1} .$$

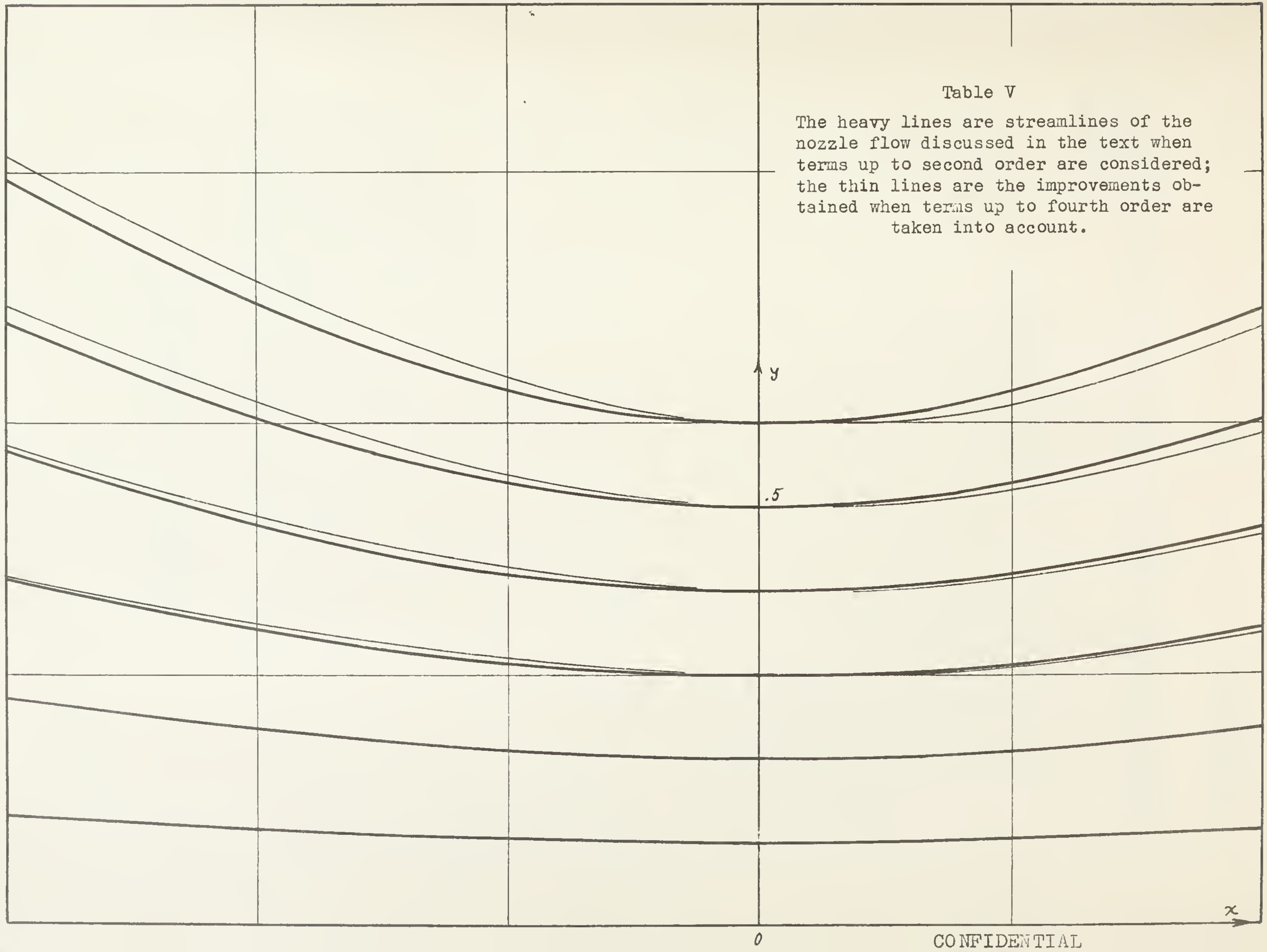


Table V

The heavy lines are streamlines of the nozzle flow discussed in the text when terms up to second order are considered; the thin lines are the improvements obtained when terms up to fourth order are taken into account.

3. Derivation of the Formulas in Appendix I of the Previous Report.

It had been suggested that derivations be given for the formulas (12.01) in Appendix I. These formulas give expressions for the derivatives of x and y with respect to ξ and η. Of course, they can be derived formally from the equations (2.01), (2.02) which give expressions for the derivatives of ϕ and with respect to x and y, observing that ϕ is a function of ξ by (2.09) and ψ a function of η by (2.10). In the following equation (12.01) is derived directly from the continuity relation and the condition that the flow be irrotational.

The two partial derivatives $\frac{\partial x}{\partial \phi}$, $\frac{\partial y}{\partial \phi}$ imply that in carrying out the differentiation, the quantity ψ should be kept constant. Since the equation ψ = const. is that of a streamline, it is clear that $\frac{\partial x}{\partial \phi}$, $\frac{\partial y}{\partial \phi}$ are derivatives along a streamline. Consequently these two quantities represent the component of a vector which is tangential to the streamline through the point in question. In other words, this vector points in the same direction as the velocity vector. The angle the velocity vector makes with the axis was denoted by θ. Let further s be the arc length along this streamline. Then we have

$$\frac{\partial x}{\partial \phi} = \frac{ds}{d\phi} \cos \theta , \quad \frac{\partial y}{\partial \phi} = \frac{ds}{d\phi} \sin \theta$$

where

$$\frac{ds}{d\phi} = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2} \ .$$

The factor $\frac{ds}{d\phi}$ can easily be evaluated. From the definition of the potential function ϕ we have

$$\frac{d\phi}{ds} = q = \sqrt{u^2 + v^2} \ ,$$

hence $\frac{ds}{d\phi} = 1/q$ and

$$\frac{\partial x}{\partial \phi} = \frac{\cos \theta}{q} \ , \quad \frac{\partial y}{\partial \phi} = \frac{\sin \theta}{q} \ .$$

When ξ is introduced as new variable instead of ϕ by $d\phi = q_o d\xi$, these relations become

$$\frac{\partial x}{\partial \xi} = \frac{q_o}{q} \cos \theta , \quad \frac{\partial y}{\partial \xi} = \frac{q_o}{q} \sin \theta \ .$$

The partial derivatives $\frac{\partial x}{\partial \psi}$, $\frac{\partial y}{\partial \psi}$ are correspondingly the components of a vector tangential to the potential line through the point in question. Hence

$$\frac{\partial x}{\partial \psi} = - \frac{ds}{d\psi} \sin \theta , \quad \frac{\partial y}{\partial \psi} = \frac{ds}{d\psi} \cos \theta \ .$$

Here s is the arc length along the potential line, and

$$\frac{ds}{d\psi} = \sqrt{\left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2} \ .$$

We assume that s increases away from the axis; then on the axis $dy = ds$; therefore the signs are correct as indicated. To evaluate $\frac{ds}{d\psi}$ we note that according to the definition of the stream function

$$d(\pi\psi^2) = \rho q d(2\pi s) \ ,$$

the left hand side being the mass flux per unit time crossing between two stream tubes $\psi =$ const. for which the difference in the values of ψ is $d\psi$. The right hand side is the product of the mass flux per unit time per unit area ρq and the area of the section of the potential surface cut out by these two stream tubes. From this relation one has

$$\psi d\psi = \rho q ds$$

or

$$\frac{ds}{d\psi} = \frac{\psi}{\rho q} \ .$$

Hence

$$\frac{\partial x}{\partial \psi} = - \frac{\psi}{\rho q} \sin\theta \ , \ \frac{\partial y}{\partial \psi} = \frac{\psi}{\rho q} \cos\theta \ .$$

When η is introduced as new variable by

$$\psi = \sqrt{\rho_* q_*} \, \eta,$$

then these equations become

$$\frac{\partial x}{\partial \eta} = -\,h^2 \sin\theta\,, \quad \frac{\partial y}{\partial \eta} = h^2 \cos\theta\,,$$

where the abbreviation

$$h^2 = \frac{\rho_* q_*}{\rho\, q}$$

has been used.

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